THE COLOUR NUMBERS OF COMPLETE GRAPHS

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1. *Preliminaries*

An *independent set of vertices* in an ordinary graph G with vertex set V and edge set X is a subset of V in which no two elements are adjacent, *i.e.*, joined by an edge. The minimum number of independent sets (also called colour classes) into which V can be partitioned is called the (*vertex*) *chromatic number* of G and is denoted by $\chi(G)$.

In a like manner, we define two other " colour numbers " for a graph G. An *independent set of edges* in G is a subset of X in which no two elements are adjacent, *i.e.*, have an end-vertex in common. The minimum number of independent sets (also called edge colour classes) into which X can be partitioned is called the *edge chromatic number* of G (also *chromatic index*, see [**1**]) and is denoted by $\chi'(G)$. An *independent set of vertices and edges* in G is a subset of $V \cup X$ having the properties: (i) no two elements of V are adjacent, (ii) no two elements of X are adjacent, (iii) no element of V is incident with an element of X. The minimum number of such independent sets into which $V \cup X$ can be partitioned is called the *total chromatic number* of G and is denoted by $\chi''(G)$.

By K_n, the *complete graph* of order n, we mean the graph where $|V|=n$ ($|V|$ denotes the cardinal number of V) and $|X|=n(n-1)/2$, *i.e.*, all distinct vertices of K_n are adjacent. A graph G with vertex set V is called *bipartite* if V can be partitioned into sets V_1 and V_2 so that every edge of G joins a vertex of V_1 to a vertex of V_2. The *complete bipartite graph* $K_{m,n}$ is the bipartite graph with $|V_1|=m$, $|V_2|=n$, and $|X|=mn$, *i.e.*, every vertex of V_1 is adjacent to all vertices of V_2.

Our purpose here is to establish the colour numbers for the complete graphs and the complete biparite graphs. Some of these results are already known, but they shall be listed for completeness.

2. *The colour numbers of K_n and $K_{m,n}$*

Before presenting our main results, we make the following observations.

Remark. If deg v denotes the degree of the vertex v in G (the number of edges in G incident with v) and $d=\max_v(\deg v)$, then $\chi'(G) \geqslant d$ and $\chi''(G) \geqslant d+1$.

THEOREM 1.

(i) $\chi(K_n)=n$;

Received 8 July, 1965.

[JOURNAL LONDON MATH. SOC., 42 (1967), 226–228]

(ii) $\chi'(K_n) = \begin{cases} n \text{ for odd } n \geqslant 3 \\ n-1 \text{ for } n \text{ even;} \end{cases}$

(iii) $\chi''(K_n) = \begin{cases} n \text{ for } n \text{ odd} \\ n+1 \text{ for } n \text{ even.} \end{cases}$

Proof. (i) It is well-known and obvious that $\chi(K_n) = n$.

(ii) We first take $n \geqslant 3$ and odd. In this case no independent set of edges can contain more than $(n-1)/2$ edges; therefore, $\chi'(K_n) \geqslant n$. Label the vertices of G 1, 2, ..., n, and let

$$S_p = \{(p-q, p+q) | q = 1, 2, \ldots, (n-1)/2\}$$

for $p = 1, 2, \ldots, n$, where for the edge $(p-q, p+q)$, each of the numbers $p-q$ and $p+q$ is expressed as one of the numbers 1, 2, ..., n modulo n. It is seen that each S_p is independent and that $\cup S_p = X$. Hence $\chi'(K_n) = n$ for odd $n \geqslant 3$.

For even n K_n is the union of $n-1$ 1-factors [**2**], consequently $\chi'(K_n) = n-1$. X can be partitioned into $n-1$ sets $T_1, T_2, \ldots, T_{n-1}$ of independent edges as follows: Assume that $n \geqslant 4$. (The theorem is obvious for $n = 2$.) Label the vertices 1, ..., n and let K_{n-1} denote the graph obtained from K_n by deleting n and all edges incident with n. For $p = 1, 2, \ldots, n-1$, let S_p be as before and let $T_p = S_p \cup \{(p, n)\}$.

(iii) As mentioned in the remark preceding this theorem, $\chi''(K_n) \geqslant n$ since $d + 1 = n$ for K_n. If n is odd, we define $S_p' = S_p \cup \{p\}$, $p = 1, 2, \ldots, n$, where S_p is as in (ii). S_p' is an independent set of vertices and edges, and every vertex and edge of K_n lies in one and only one S_p'; hence, $\chi''(K_n) = n$ for n odd.

We now consider the case where n is even. Note that $|V \cup X| = n(n+1)/2$ and that no independent set of vertices and edges can contain more than $n/2$ elements since such a set contains at most one vertex. This implies that for n even, $\chi''(K_n) \geqslant n+1$. To K_n we add a vertex labelled $n+1$ and add the edges $(j, n+1)$, $j = 1, 2, \ldots, n$, producing the graph K_{n+1}. Since $n+1$ is odd, $\chi''(K_{n+1}) = n+1$ and because K_n is a subgraph of K_{n+1}, we clearly have $\chi''(K_n) \leqslant \chi''(K_{n+1})$. Therefore, $\chi''(K_n) = n+1$.

In the next theorem, we make use of the Kronecker delta δ_{mm}, which has the value 1 or 0 depending on whether $m = n$ or $m \neq n$.

THEOREM 2.

(i) $\chi(K_{m,n}) = 2$;

(ii) $\chi'(K_{mn}) = \max(m, n)$;

(iii) $\chi''(K_{m,n}) = \max(m, n) + 1 + \delta_{mn}$.

Proof. (i) is obvious.

(ii) follows from König's theorem that for any bipartite graph G $\chi'(G) = d$ (see [3]). (ii) can also be proved simply as follows: **Let** $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ and suppose that $m \leqslant n$.

Let $A_p = \{(u_i, v_{i+p}) | i = 1, 2, \ldots, m\}$ for $p = 1, 2, \ldots, n$, the suffixes being taken mod n. The edges of $K_{m,n}$ are partitioned into the n independent sets $A_1, A_2, \ldots, A_n$.

(iii) Assume $m < n$. By the remark preceding Theorem 1, $\chi''(K_{m,n}) \geqslant n+1$. For $p = 1, 2, \ldots, n$, define $B_p = A_p \cup \{v_p\}$, where A_p is as defined above and $B_{n+1} = V_1$. The sets B_p, $p = 1, 2, \ldots, n+1$, are independent sets of vertices and edges which partition $V \cup X$. Therefore, $\chi''(K_{m,n}) \leqslant n+1$ so that $\chi''(K_{m,n}) = n+1$ for $m < n$.

If $m = n$, we note that no independent set of vertices and edges in $K_{n,n}$ can contain more than n elements and since $|V \cup X| = n^2 + 2n$, this implies that $\chi''(K_{n,n}) \geqslant n+2$. However, if we define A_p as before for $p = 1, 2, \ldots, n$, set $A_{n+1} = V_1$, and let $A_{n+2} = V_2$, we then have a partitioning of $V \cup X$ into $n+2$ independent sets, and so $\chi''(K_{n,n}) = n+2$.

3. *Additional remarks*

It is clear that for any graph G, $\chi(G) + \chi'(G) \geqslant \chi''(G)$. For the graph $K_{n,n}$, we found that $\chi(K_{n,n}) + \chi'(K_{n,n}) = \chi''(K_{n,n})$. We shall now show equality holds only if the graph is bipartite.

THEOREM 3. *If G has at least two points and $\chi(G) + \chi'(G) = \chi''(G)$, then G is bipartite.*

Proof. Assume $\chi(G) + \chi'(G) = \chi''(G)$ and that G is not bipartite so that $\chi(G) \geqslant 3$. Let $\cup U_r$ be a minimal partition of V into (3 or more) independent sets, and let $\cup S_p$ be a minimal partition of X into independent sets. Now $(\cup U_r) \cup (\cup S_p)$ is a partition of $V \cup X$ into independents sets, but we shall show that this is not a minimal partition. Since $\chi(G) \geqslant 3$, for each edge in S_1, say, there exists some U_r which has none of its elements incident with the edge. If each edge of S_1 is added to such a U_r, then the resulting sets are independent sets of vertices and edges. This then implies that the set S_1 may be deleted and a smaller partition of $V \cup X$ into independent sets can be found; therefore, $\chi(G) + \chi'(G) > \chi''(G)$, but this is a contradiction.

The converse of Theorem 3 is not true as can be seen by considering $K_{m,n}$, $m \neq n$, $K_{3,3}$, for example.

References

1. D. König, *Theorie der endlichen und unendlichen Graphen* (Leipzig, 1936), p. 171.
2. ———, *loc. cit.* p. 157, Satz 2.
3. ———, *loc. cit.* p. 171, Satz 15.

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